IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

A hierarchy of coupled Burgers systems possessing a hereditary structure

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1993 J. Phys. A: Math. Gen. 26 L1169

(http://iopscience.iop.org/0305-4470/26/22/007)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.68
The article was downloaded on 01/06/2010 at 20:02

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 26 (1993) L1169–L1174. Printed in the UK

LETTER TO THE EDITOR

# A hierarchy of coupled Burgers systems possessing a hereditary structure

Wen-xiu Ma

CCAST (World Laboratory) PO Box 8730, Beijing, 100080, People's Republic of China and Institute of Mathematics, Fudan University, Shanghai 200433, People's Republic of China†

Received 1 September 1993

**Abstract.** A hierarchy of typical integrable, coupled Burgers systems is proposed by introducing a special spectral problem involving $q$ dependent variables $u_0, u_1, \ldots, u_{q-1}$. These systems possess a hereditary structure and may reduce to a hierarchy of Burgers equations under the reduction $u_i=0$, $0\leqslant i\leqslant q-2$. It is shown that their flows and Lax operators commute mutually but each system is not Hamiltonian.

After the famous AKNS paper [1], many nonlinear systems of evolution equations integrable by the use of the inverse scattering transform techniques have been presented. Part of these systems possess Hamiltonian structures and an infinite number of symmetries and conserved densities. Moreover, the symmetries and the conserved densities are related to each other through the Hamiltonian structures. This class of integrable systems is very large and appears in various fields [2, 3]. However, there also exist nonlinear integrable systems which do not possess Hamiltonian structures but have an infinite number of symmetries. For example, Burgers equation $u_t=u_{xx}+uu_x$. They belong to the other class of integrable systems in which only a few examples are found.

In the present letter, we would like to provide new integrable systems for the second class. We propose a hierarchy of coupled Burgers systems from a special spectral problem. This hierarchy possesses a hereditary structure and may reduce to a hierarchy of Burgers equations under some reduction, which is similar to coupled KdV systems [4] and coupled Harry–Dym systems [5]. But here Lax pairs cannot give a construction of the associated Hamiltonian structures and Hamiltonian functions. Moreover, we fail to construct Darboux transformations [6] of the corresponding hierarchy.

Let us now consider the spectral problem with the potential $u=(u_0, u_1, \ldots, u_{q-1})^T$:

$$\phi_x=U\phi=U(u,\lambda)\phi \qquad \phi=\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \qquad U=\begin{bmatrix}1 & A+1\\ A-1 & -1\end{bmatrix}=\sigma_3+i\sigma_2+A\sigma_1 \tag{1}$$

where $A=u_0\lambda^{-(q-1)}+u_1\lambda^{-(q-2)}+\ldots+u_{q-2}\lambda^{-1}+u_{q-1}-\lambda$, $q\geqslant 1$, and the $\sigma_j$, $1\leqslant j\leqslant 3$, are $2\times 2$ Pauli matrices. This spectral problem is found while studying coupled KdV systems. The spectral problem (1) is a particular case of the general scheme with an $SL(2, C)$ spectral operator depending polynomially on $\lambda$ and $\lambda^{-1}$. Interestingly, associated with the spectral problem (1), there exists a hierarchy of coupled Burgers systems,

† Mailing address.

0305-4470/93/221169+06$07.50 © 1993 IOP Publishing Ltd 

which possesses a hereditary structure. Usually, the search for new hereditary symmetries [7] is a difficult thing. Here, by the way, we present a new hereditary symmetry $\Phi$. In what follows, we exhibit carefully the concrete construction process.

According to the generating scheme in [8], we first find a solution to the adjoint representation equation $V_x = [U, V]$ of (1). Set

$$V = V(u, \lambda) = \begin{bmatrix} B_1 & B_2 \\ B_3 & -B_1 \end{bmatrix} = \sum_{i=-1}^{\infty} V_i \lambda^{-i}. \tag{2}$$

At this moment we see that $V_x = [U, V]$ is equivalent to

$$B_{1x} = -A(B_2 - B_3) + (B_2 + B_3) \tag{3a}$$

$$B_{2x} = 2B_2 - 2AB_1 - 2B_1 \tag{3b}$$

$$B_{3x} = 2AB_1 - 2B_1 - 2B_3. \tag{3c}$$

The equations (3*b*) and (3*c*) lead to

$$(B_2 + B_3)_x = -4B_1 + 2(B_2 - B_3) \tag{4a}$$

$$(B_2 - B_3)_x = -4AB_1 + 2(B_2 + B_3). \tag{4b}$$

By comparing (3*a*) with (4*b*), we can make a possible choice

$$B_1 = \tfrac{1}{2}(B_2 - B_3) \qquad B_2 + B_3 = B_{1x} + 2AB_1$$

or equivalently

$$B_2 = \tfrac{1}{2}B_{1x} + (A+1)B_1 \qquad B_3 = \tfrac{1}{2}B_{1x} + (A-1)B_1. \tag{5}$$

Now (4*a*) requires a constraint

$$(B_2 + B_3)_x = 2(\tfrac{1}{2}\partial^2 + \partial A)B_1 = 0. \tag{6}$$

We can further assume that

$$B_1 = \sum_{i=0}^{\infty} b_i \lambda^{-i} = \sum_{i=0}^{\infty} b_i[u] \lambda^{-i}.$$

Then we can calculate that

$$(\tfrac{1}{2}\partial + A)B_1 = \sum_{k \leqslant q} (u_0, u_1, \ldots, u_{q-2}, u_{q-1} + \tfrac{1}{2}\partial, -1) \begin{bmatrix} b_{-k} \\ b_{-k+1} \\ \vdots \\ b_{-k+(q-2)} \\ b_{-k+(q-1)} \\ b_{-k+q} \end{bmatrix} \lambda^{k-(q-1)}$$

$$= -b_0\lambda + \sum_{k \leqslant q-1} (u_0, u_1, \ldots, u_{q-2}, u_{q-1} + \tfrac{1}{2}\partial, -1) \begin{bmatrix} b_{-k} \\ b_{-k+1} \\ \vdots \\ b_{-k+(q-2)} \\ b_{-k+(q-1)} \\ b_{-k+q} \end{bmatrix} \lambda^{k-(q-1)}$$

$$:= \sum_{k \leqslant q} c_k \lambda^{k-(q-1)}. \tag{7}$$

Here and in the following we suppose that $b_i=0$, $i<0$. To obtain a solution of (6), we choose $c_q=\text{constant}$, $c_i=0$, $i\leqslant q-1$, by which we obtain a recursive formula for $b_i$:

$$b_0=\alpha=\text{constant},\ b_i=u_0b_{i-q}+u_1b_{i-(q-1)}+\ldots+u_{q-1}b_{i-1}+\tfrac{1}{2}b_{i-1,x} \qquad i\geqslant 1. \tag{8}$$

The functions $b_i$ are all polynomials in $u, u_x, \ldots$. For example, we have

$$b_1=\alpha u_{q-1} \qquad b_2=\alpha u_{q-2}+\alpha u_{q-1}^2+\tfrac{1}{2}\alpha u_{q-1,x}.$$

By now we acquire a solution of $V_x=[U, V]$:

$$\begin{aligned} V=V(u,\lambda)&=\begin{bmatrix} B_1 & \tfrac{1}{2}B_{1x}+(A+1)B_1 \\ \tfrac{1}{2}B_{1x}+(A-1)B_1 & -B_1 \end{bmatrix} \\ &=B_1U+\tfrac{1}{2}B_{1x}\sigma_1 \end{aligned} \tag{9}$$

with

$$B_1=\sum_{i=0}^{\infty} b_i\lambda^{-i}$$

where $b_i$s are determined by (8).

Next we discuss Lax pairs and the corresponding integrable systems. According to [8], let us choose

$$V^{(m)}=V^{(m)}(\lambda)=(\lambda^mV)_++\Delta_m=\begin{bmatrix} V_1^{(m)}(\lambda) & V_2^{(m)}(\lambda) \\ V_3^{(m)}(\lambda) & -V_1^{(m)}(\lambda) \end{bmatrix} \tag{10}$$

where $V$ is given by (9), the $\Delta_m=\Delta_m(u,\lambda)$ are supposed to be a Laurent polynomial form of $\lambda$ and the sign + denotes the selection of non-negative powers of $\lambda$. Similarly, the zero curvature conditions

$$U_t-V_x^{(m)}+[U, V^{(m)}]=0 \qquad m\geqslant 0$$

of Lax pairs $\phi_x=U\phi$, $\phi_t=V^{(m)}\phi$, $m\geqslant 0$, may engender that

$$V_1^{(m)}(\lambda)=(\lambda^mB_1)_+=\sum_{i=0}^{m} b_i\lambda^{m-i} \qquad m\geqslant 0 \tag{11a}$$

$$V_2^{(m)}(\lambda)=\tfrac{1}{2}V_{1x}^{(m)}(\lambda)+(A+1)V_1^{(m)}(\lambda) \qquad m\geqslant 0 \tag{11b}$$

$$V_3^{(m)}(\lambda)=\tfrac{1}{2}V_{1x}^{(m)}(\lambda)+(A-1)V_1^{(m)}(\lambda) \qquad m\geqslant 0 \tag{11c}$$

and the constraint

$$A_t=(\tfrac{1}{2}\partial^2+\partial A)V_1^{(m)}(\lambda) \qquad m\geqslant 0. \tag{12}$$

The equality (11) means just that the modified quantities $\Delta_m$ take the form

$$\Delta_m=(\lambda^mB_1)_+U-(\lambda^mB_1U)_+ \qquad m\geqslant 0. \tag{13}$$

Further by using (8), we have

$$(\tfrac{1}{2}\partial+A)(\lambda^mB_1)_+=\sum_{i=0}^{q-2}\left(\sum_{j=0}^{i} u_jb_{j+m-i}\right)\lambda^{i-(q-1)}+b_{m+1}-\alpha\lambda^{m+1}.$$

Thus by (12) and (11*a*), we acquire a hierarchy of systems of evolution equations

$$u_t = K_m = \partial\left(u_0 b_m, u_0 b_{m-1} + u_1 b_m, \ldots, \sum_{j=0}^{q-2} u_j b_{j+m-(q-2)}, b_{m+1}\right)^T \qquad m \geqslant 0 \tag{14}$$

which possess Lax operators

$$V^{(m)} = \begin{bmatrix} (\lambda^m B_1)_+ & \frac{1}{2}(\lambda^m B_{1x})_+ + (A+1)(\lambda^m B_1)_+ \\ \frac{1}{2}(\lambda^m B_{1x})_+ + (A-1)(\lambda^m B_1)_+ & -(\lambda^m B_1)_+ \end{bmatrix}$$

$$= (\lambda^m B_1)_+ U + \tfrac{1}{2}(\lambda^m B_{1x})_+ \sigma_1 \qquad m \geqslant 0. \tag{15}$$

We observe (14) a little more and then find that $K_m = \Phi K_{m-1}, m \geqslant 0$, where the operator $\Phi$ reads as

$$\Phi = \begin{bmatrix} 0 & 0 & \ldots & 0 & P_0 \\ 1 & 0 & \ldots & 0 & P_1 \\ 0 & 1 & \ldots & 0 & P_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & P_{q-1} \end{bmatrix} \tag{16}$$

with $P_i = \partial u_i \partial^{-1}, 0 \leqslant i \leqslant q-2, P_{q-1} = \frac{1}{2}\partial + \partial u_{q-1}\partial^{-1}$. Therefore (14) may be written as

$$u_t = K_m = \Phi K_{m-1} = \ldots = \Phi^m K_0 = \Phi^m(\alpha u_x) \qquad m \geqslant 0. \tag{17}$$

A direct computation can give that $\Phi$ is a hereditary symmetry [7]:

$$\Phi^2[K, S] + [\Phi K, \Phi S] - \Phi\{[K, \Phi S] + [\Phi K, S]\} = 0 \tag{18}$$

where the commutator of vector fields is defined as $[K, S] = K'[S] - S'[K]$. Therefore (14) possesses a hereditary structure. Besides, we easily find that the Lie derivative of $\Phi$ with respect to $K_0 = \alpha u_x$ takes zero, i.e.

$$L_{K_0}\Phi = \Phi'[K_0] - [K_0', \Phi] = 0. \tag{19}$$

Hence the tensor operator $\Phi = \Phi(u)$ is to be invariant along the trajectories of the vector field $K_0$ [9]. Now according to the result of [10] or [7, 11, 12], it follows from (18) and (19) that $\Phi$ is a common recursion operator (or strong symmetry) of the whole hierarchy (17) and that

$$[K_m, K_n] = [\Phi^m K_0, \Phi^n K_0] = (m-n)(L_{K_0}\Phi)\Phi^{m+n-1}K_0 = 0 \qquad m, n \geqslant 0. \tag{20}$$

It follows that the flows of the hierarchy (17) commute mutually and that each system in the hierarchy (17) possesses an infinite number of symmetries $\{K_m\}_{m=0}^{\infty}$.

Let us recall the products of Lax operators $V^{(m)}$ proposed in [13]:

$$[\![V^{(m)}, V^{(n)}]\!] = V^{(m)\prime}[K_n] - V^{(n)\prime}[K_m] + [V^{(m)}, V^{(n)}] \qquad m, n \geqslant 0. \tag{21}$$

Through the general algebraic structure of zero curvature representations in [13], we see that

$$U'[\![K_m, K_n]\!] - [\![V^{(m)}, V^{(n)}]\!]_x + [U, [\![V^{(m)}, V^{(n)}]\!]] = 0 \qquad m, n \geqslant 0. \tag{22}$$

Therefore from (20), we find that the $[\![V^{(m)}, V^{(n)}]\!]$, $m, n \geqslant 0$, satisfy the adjoint representation equation

$$[\![V^{(m)}, V^{(n)}]\!]_x = [U, [\![V^{(m)}, V^{(n)}]\!]] \qquad m, n \geqslant 0. \tag{23}$$

In addition, we can immediately verify that for the spectral operator $U$ given by (1), if a matrix $V=V(u,\lambda)$ depending polynomially on $\lambda$ and $\lambda^{-1}$ satisfies the adjoint representation equation $V_x=[U,V]$ and $V|_{u=0}=0$, then we have $V=0$. This property is universally applicable to only a few spectral problems. Now by the property, we obtain a commutative Lax operator algebra

$$[\![V^{(m)},V^{(n)}]\!]=0 \qquad m,n\geqslant 0. \tag{24}$$

Conversely, we may first show (24) by direct calculation. Then from (22) and the injection of $U'$, we may also obtain the commutative property of flows of (17). This is an application of Lax operator algebras [14]. In fact, (24) implies the commutative property of flows.

Making the reduction $u_i=0$, $0\leqslant i\leqslant q-2$, we see that (17) reduces to

$$u_{q-1,t}=P^m_{q-1}(\alpha u_{q-1,x}) \qquad m\geqslant 0 \tag{25}$$

which is exactly a hierarchy of Burgers equations. Therefore we call (17) coupled Burgers systems. Particularly when $q=1$, we obtain only a hierarchy of Burgers equations. Because (25) does not possess Hamiltonian structures, the hierarchy (17) belongs to the second class of typical integrable systems. The first nonlinear system of (17) reads as

$$u_t=\Phi(\alpha u_x)=\alpha\begin{bmatrix}(u_0u_{q-1})_x\\ u_{0x}+(u_1u_{q-1})_x\\ \vdots\\ u_{q-3,x}+(u_{q-2}u_{q-1})_x\\ u_{q-2,x}+\frac{1}{2}u_{q-1,xx}+2u_{q-1}u_{q-1,x}\end{bmatrix}. \tag{26}$$

This system can be expressed as

$$u_t=\alpha JG_0=\alpha\begin{bmatrix}0&\dots&0&\partial\\ 0&\dots&\partial&0\\ \vdots&\ddots&\vdots&\vdots\\ \partial&\dots&0&0\end{bmatrix}\begin{bmatrix}u_{q-2}+\frac{1}{2}u_{q-1,x}+u^2_{q-1}\\ u_{q-3}+u_{q-2}u_{q-1}\\ \vdots\\ u_0+u_1u_{q-1}\\ u_0u_{q-1}\end{bmatrix}.$$

Here $J$ is a Hamiltonian operator. However, $(G_0')^*\neq G_0'$ and thus $G_0$ is not a gradient vector field. This also shows that the system (26) does not have local Hamiltonian structures.

This work was supported by the National Natural Science Foundation of China and the Shanghai Science and Technology Committee of China.

## References

[1] Ablowitz M J, Kaup D J, Newell A C and Segur H 1974 *Stud. Appl. Math.* **53** 249
[2] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia: SIAM)
[3] Newell A C 1985 *Solitons in Mathematics and Physics* (Philadelphia: SIAM)
[4] Antonowicz M and Fordy A P 1987 *Physica* **28D** 345
[5] Antonowicz M and Fordy A P 1988 *J. Phys. A: Math. Gen.* **21** L269
[6] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[7] Fuchssteiner B 1979 *Nonlinear Analysis TMA* **3** 849

[8] Tu G Z 1989 *J. Phys. A: Math. Gen.* **22** 2375
[9] Magri F 1980 *Nonlinear Evolution Equations and Dynamical Systems* Lecture Notes in Physics vol 120 ed M Boiti, F Pempinelli and G Soliani (Berlin: Springer) p 233
[10] Ma W X 1990 *J. Phys. A: Math. Gen.* **23** 2707
[11] Oevel W 1987 *Topics in Soliton Theory and Exactly Solvable Nonlinear Equations* ed M Ablowitz, B Fuchssteiner and M Kruskal (Singapore: World Scientific) p 108
[12] Fokas A S and Santini P M 1988 *Symmetries and Nonlinear Phenomena* ed D Levi and P Winternitz (Singapore: World Scientific) p 7
[13] Ma W X 1993 *J. Phys. A: Math. Gen.* **26** to appear
[14] Ma W X 1992 *J. Math. Phys.* **33** 2464